Einstein Field Equations (EFE)

1 - General Relativity Origins

In the 1910s, Einstein studied gravity. Following the reasoning of Faraday and Maxwell, he thought that if two objects are attracted to each other, there would be some medium. The only medium he knew in 1910 was spacetime. He then deduced that the gravitational force is an indirect effect carried by spacetime. He concluded that any mass perturbs spacetime, and that the spacetime, in turn, has an effect on mass, which is gravitation. So, when an object enters in the volume of the curvature of spacetime made by a mass, i.e. the volume of a gravitational field, it is subject to an attracting force. In other words, Einstein assumed that the carrier of gravitation is the *curvature of spacetime*. Thus, he tried to find an equation connecting:

1. The curvature of spacetime. This mathematical object, called the "Einstein tensor", is the left hand side of the EFE (Eq. 1).
2. The properties of the object that curves spacetime. This quantity, called the "Energy–Momentum tensor", is $T_{\mu\nu}$ in the right hand side of the EFE.

$$\textit{Curvature of spacetime} \quad \equiv \quad \textit{Object producing this curvature}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (1)$$

Einstein Tensor (curvature of spacetime)

Thus, Einstein early understood that gravity is a consequence of the curvature of spacetime. Without knowing the mechanism of this curvature, he posed the question of the special relativity in a curved space. He left aside the flat Minkowski space to move to a Gaussian curved space. The latter leads to a more general concept, the "Riemannian space". On the other hand, he identified the gravitational acceleration to the inertial acceleration (see the Appendix G "New Version of the Equivalent Principle").

The curvature of a space is not a single number, though. It is described by "tensors", which are a kind of matrices. For a 4D space, the curvature is given by the Riemann-Christoffel tensor which becomes the Ricci Tensor after reductions. From here, Einstein created another tensor called "Einstein Tensor" (left hand side of equation 1) which combines the Ricci Curvature Tensor $R_{\mu\nu}$, the metric tensor $g_{\mu\nu}$ and the scalar curvature R (see the explanations below).

Fluid Mechanics

In fluid mechanics, the medium has effects on objects. For example, the air (the medium) makes pressures on airplanes (objects), and also produces perturbations around them. So, Einstein thought that the fluid mechanics could be adapted to gravity. He found that the Cauchy-Stress Tensor was close to what he was looking for. Thus, he identified 1/ the **"volume"** in fluid mechanics to **"mass"**, and 2/ the **"fluid"** to **"spacetime"**.

Energy-Momentum Tensor

The last thing to do is to include the characteristics of the object that curves spacetime in the global formulation. To find the physical equation, Einstein started with the elementary volume $dx.dy.dz$ in fluid mechanics. The tensor that describes the forces on the surface of this elementary volume is the Cauchy Tensor, often called Stress Tensor. However, this tensor is in 3D. To convert it to 4D, Einstein used the "Four-Vectors" in Special Relativity. More precisely, he used the "Four-Momentum" vector Px, Py, Pz and Pt. The relativistic "Four-Vectors" in 4D (x, y, z and t) are an extension of the well-known non-relativistic 3D (x, y and z) spatial vectors. Thus, the original 3D Stress Tensor of the fluid mechanics became the 4D Energy-Momentum Tensor of EFE.

Einstein Field Equations (EFE)

Finally, Einstein identified its tensor that describes the curvature of spacetime to the Energy-Momentum tensor that describes the characteristics of the object which curves spacetime. He added an empirical coefficient, $8\pi G/c^4$, to homogenize the right hand side of the EFE. Today, this coefficient is calculated to get back the Newton's Law from EFE in the case of a static sphere in a weak field. If no matter is present, the energy-momentum tensor vanishes, and we come back to a flat spacetime without gravitational field.

The Proposed Theory

However, some unsolved questions exist in the EFE, despite the fact that they work perfectly. For example, Einstein built the EFE without knowing 1/ what is mass, 2/ the mechanism of gravity, 3/ the mechanism by which spacetime is curved by mass . . . To date, these enigmas remain. Considering that "mass curves spacetime" does not explain anything. No one knows by which strange phenomenon a mass can curve spacetime. It seems obvious that *if a process makes a deformation of spacetime, it may reasonably be expected to provide information about the nature of this phenomenon*. Therefore, the main purpose of the present paper is to try to solve these enigmas, i.e. to give a rational explanation of mass, gravity and spacetime curvature. The different steps to achieve this goal are:

- ***Special Relativity (SR).*** This section gives an overview of SR.
- ***Einstein Tensor.*** Explains the construction of the Einstein Tensor.
- ***Energy-Momentum Tensor.*** Covers the calculus of this tensor.
- ***Einstein Constant.*** Explains the construction of the Einstein Constant.
- ***EFE.*** This section assembles the three precedent parts to build the EFE.

2 - Special Relativity (Background)

Lorentz Factor

$$\beta = \frac{v}{c} \tag{2}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}} \tag{3}$$

Minkowski Metric with signature $(-, +, +, +)$:

$$\eta_{\mu\nu} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{5}$$

Minkowski Metric with signature $(+, -, -, -)$:

$$\eta_{\mu\nu} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{6}$$

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{7}$$

Time Dilatation
"τ" is the proper time.

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2d\tau^2 \tag{8}$$

$$d\tau^2 = \frac{ds^2}{c^2} \quad \Rightarrow \quad d\tau^2 = dt^2\left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}\right) \tag{9}$$

$$v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} \tag{10}$$

$$d\tau = dt\ \sqrt{1 - \beta^2} \tag{11}$$

Lenght Contractions

"dx'" is the proper lenght.

$$dx' = \frac{dx}{\sqrt{1-\beta^2}} \tag{12}$$

Lorentz Transformation

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned} \tag{13}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix} \tag{14}$$

Inverse transformation on the x-direction:

$$\begin{aligned} ct &= \gamma(ct' + \beta x') \\ x &= \gamma(x' + \beta ct') \\ y &= y' \\ z &= z' \end{aligned} \tag{15}$$

$$\begin{bmatrix} c\,t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,t' \\ x' \\ y' \\ z' \end{bmatrix} \tag{16}$$

Four-Position

Event in a Minkowski space:

$$\mathbf{X} = x_\mu = (x_0, x_1, x_2, x_3) = (ct, x, y, z) \tag{17}$$

Displacement:

$$\Delta X_\mu = (\Delta x_0, \Delta x_1, \Delta x_2, \Delta x_3) = (c\Delta t, \Delta x, \Delta y, \Delta z) \tag{18}$$

$$dx_\mu = (dx_0, dx_1, dx_2, dx_3) = (cdt, dx, dy, dz) \tag{19}$$

Four-Velocity

v_x, v_y, v_z = Traditional speed in 3D.

$$\mathbf{U} = u_\mu = (u_0, u_1, u_2, u_3) = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) \qquad (20)$$

from (7):

$$ds^2 = c^2 dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2} \right) \qquad (21)$$

$$v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2} \qquad (22)$$

$$ds^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2} \right) \quad \Rightarrow \quad ds = cdt\ \sqrt{1 - \beta^2} \qquad (23)$$

Condensed form:

$$u_\mu = \frac{dx_\mu}{d\tau} = \frac{dx_\mu}{dt}\frac{dt}{d\tau} \qquad (24)$$

Thus:

$$u_0 = \frac{dx_0}{d\tau} = \frac{cdt}{dt\ \sqrt{1 - \beta^2}} = \frac{c}{\sqrt{1 - \beta^2}} = \gamma c \qquad (25)$$

$$u_1 = \frac{dx_1}{d\tau} = \frac{dx}{dt\ \sqrt{1 - \beta^2}} = \frac{v_x}{\sqrt{1 - \beta^2}} = \gamma v_x \qquad (26)$$

$$u_2 = \frac{dx_2}{d\tau} = \frac{dy}{dt\ \sqrt{1 - \beta^2}} = \frac{v_y}{\sqrt{1 - \beta^2}} = \gamma v_y \qquad (27)$$

$$u_3 = \frac{dx_3}{d\tau} = \frac{dz}{dt\ \sqrt{1 - \beta^2}} = \frac{v_z}{\sqrt{1 - \beta^2}} = \gamma v_z \qquad (28)$$

Four-Acceleration

$$a_i = \frac{du_i}{d\tau} = \frac{d^2 x_i}{d\tau^2} \qquad (29)$$

Four-Momentum

p_x, p_y, p_z = Traditional momentum in 3D
U = Four-velocity

$$\mathbf{P} = m\mathbf{U} = m(u_0, u_1, u_2, u_3) = \gamma(mc, p_x, p_y, p_z) \qquad (30)$$

(p = mc = E/c)

$$\|\mathbf{P}\|^2 = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2c^2 \qquad (31)$$

$$p_\mu = mu_\mu = m\frac{dx_\mu}{d\tau} \qquad (32)$$

hence

$$p_0 = mc\frac{dt}{d\tau} = mu_0 = \gamma mc = \gamma E/c \qquad (33)$$

$$p_1 = m\frac{dx}{d\tau} = mu_1 = \gamma mv_x \qquad (34)$$

$$p_2 = m\frac{dy}{d\tau} = mu_2 = \gamma mv_y \qquad (35)$$

$$p_3 = m\frac{dz}{d\tau} = mu_3 = \gamma mv_z \qquad (36)$$

Force

$$\mathbf{F} = F_\mu = \frac{dp_\mu}{d\tau} = \left(\frac{d(mu_0)}{d\tau}, \frac{d(mu_1)}{d\tau}, \frac{d(mu_2)}{d\tau}, \frac{d(mu_3)}{d\tau}\right) \qquad (37)$$

3 - Einstein Tensor

Since the Einstein Tensor is not affected by the presented theory, one could think that it is not useful to study it in the framework of this document. However, the knowledge of the construction of the Einstein Tensor is necessary to fully understand the four inconsistencies highlighted and solved in this document. Therefore, this section is only a summary of the Einstein Tensor. A more accurate development of EFE can be obtained on books or on the Internet.

The Gauss Coordinates

Consider a curvilinear surface with coordinates u and v (Fig. 1A). The distance between two points, $M(u, v)$ and $M'(u + du, v + dv)$, has been calculated by Gauss. Using the g_{ij} coefficients, this distance is:

$$ds^2 = g_{11}du^2 + g_{12}dudv + g_{21}dvdu + g_{22}dv^2 \qquad (38)$$

The Euclidean space is a particular case of the Gauss Coordinates that reproduces the Pythagorean theorem (Fig. 1B). In this case, the Gauss coefficients are $g_{11} = 1$, $g_{12} = g_{21} = 0$, and $g_{22} = 1$.

$$ds^2 = du^2 + dv^2 \qquad (39)$$

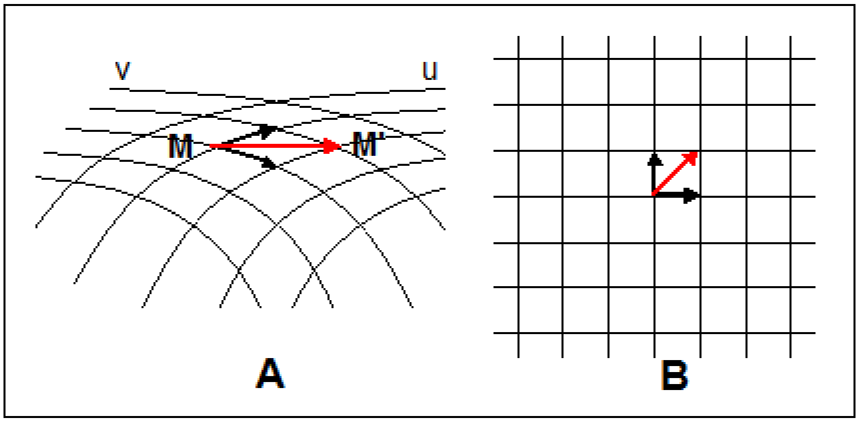

Figure 1: Gauss coordinates in a curvilinear space (A) and in an Euclidean space (B).

Equation (39) may be condensed using the Kronecker Symbol δ, which is 0 for $i \neq j$ and 1 for $i = j$, and replacing du and dv by du_1 and du_2. For indexes $i, j = 0$ and 1, we have:

$$ds^2 = \delta_{ij} du_i du_j \qquad (40)$$

The Metric Tensor

Generalizing the Gaussian Coordinates to "n" dimensions, equation (38) can be rewritten as:

$$ds^2 = g_{\mu\nu} du_\mu du_\nu \qquad (41)$$

or, with indexes μ and ν that run from 1 to 3 (example of x, y and z coordinates):

$$ds^2 = g_{11} du_1^2 + g_{12} du_1 du_2 + g_{21} du_2 du_1 \cdots + g_{32} du_3 du_2 + g_{33} du_3^2 \qquad (42)$$

This expression is often called the "Metric" and the associated tensor, $g_{\mu\nu}$, the "Metric Tensor". In the spacetime manifold of RG, μ and ν are indexes which run from 0 to 3 (t, x, y and z). Each component can be viewed as a multiplication factor which must be placed in front of the differential displacements. Therefore, the matrix of coefficients $g_{\mu\nu}$ are a tensor 4×4, i.e. a set of 16 real-valued functions defined at all points of the spacetime manifold.

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} \qquad (43)$$

However, in order for the metric to be symmetric, we must have:

$$g_{\mu\nu} = g_{\nu\mu} \qquad (44)$$

...which reduces to 10 independent coefficients, 4 for the diagonal in bold face in equation (45), g_{00}, g_{11}, g_{22}, g_{33}, and 6 for the half part above - or under - the diagonal, i.e. $g_{01} = g_{10}$, $g_{02} = g_{20}$, $g_{03} = g_{30}$, $g_{12} = g_{21}$, $g_{13} = g_{31}$, $g_{23} = g_{32}$. This gives:

$$g_{\mu\nu} = \begin{bmatrix} \mathbf{g_{00}} & g_{01} & g_{02} & g_{03} \\ (g_{10} = g_{01}) & \mathbf{g_{11}} & g_{12} & g_{13} \\ (g_{20} = g_{02}) & (g_{21} = g_{12}) & \mathbf{g_{22}} & g_{23} \\ (g_{30} = g_{03}) & (g_{31} = g_{13}) & (g_{32} = g_{23}) & \mathbf{g_{33}} \end{bmatrix} \qquad (45)$$

To summarize, the metric tensor $g_{\mu\nu}$ in equations (43) and (45) is a matrix of functions which tells how to compute the distance between any two points in a given space. The metric components obviously depend on the chosen local coordinate system.

The Riemann Curvature Tensor

The Riemann curvature tensor $R^{\alpha}_{\beta\gamma\delta}$ is a four-index tensor. It is the most standard way to express curvature of Riemann manifolds. In spacetime, a 2-index tensor is associated to each point of a 2-index Riemannian manifold. For example, the Riemann curvature tensor represents the force experienced by a rigid body moving along a geodesic.

The Riemann tensor is the only tensor that can be constructed from the metric tensor and its first and second derivatives. These derivatives must exist if we are in a Riemann manifold. They are also necessary to keep homogeneity with the right hand side of EFE which can have first derivative such as the velocity dx/dt, or second derivative such as an acceleration d^2x/dt^2.

Christoffel Symbols

The Christoffel symbols are tensor-like objects derived from a Riemannian metric $g_{\mu\nu}$. They are used to study the geometry of the metric. There are two closely related kinds of Christoffel symbols, the first kind Γ_{ijk}, and the second kind Γ^k_{ij}, also known as "affine connections" or "connection coefficients".

At each point of the underlying n-dimensional manifold, the Christoffel symbols are numerical arrays of real numbers that describe, in coordinates, the effects of parallel transport in curved surfaces and, more generally, manifolds. The Christoffel symbols may be used for performing practical calculations in differential geometry. In particular, the Christoffel symbols are used in the construction of the Riemann Curvature Tensor.

In many practical problems, most components of the Christoffel symbols are equal to zero, provided the coordinate system and the metric tensor possesses some common symmetries.

Comma Derivative

The following convention is often used in the writing of Christoffel Symbols. The components of the gradient dA are denoted $A_{,k}$ (a comma is placed before the index) and are given by:

$$A_{,k} = \frac{\partial A}{\partial x^k} \qquad (46)$$

Christoffel Symbols in spherical coordinates

The best way to understand the Christoffel symbols is to start with an example. Let's consider vectorial space $\mathbb{E}_3$ associated to a punctual space in spherical coordinates $\mathcal{E}_3$. A vector **OM** in a fixed Cartesian coordinate system $(0, e_i^0)$ is defined as:

$$\mathbf{OM} = x^i e_i^0 \qquad (47)$$

or

$$\mathbf{OM} = r\ sin\theta\ cos\varphi\ e_1^0 + r\ sin\theta\ sin\varphi\ e_2^0 + r\ cos\theta\ e_3^0 \qquad (48)$$

Calling e_k the evolution of **OM**, we can write:

$$e_k = \partial_k (x^i e_i^0) \qquad (49)$$

We can calculate the evolution of each vector e_k. For example, the vector e_1 (equation 50) is simply the partial derivative regarding r of equation (48). It means that the vector e_1 will be supported by a line OM oriented from $zero$ to infinity. We can calculate the partial derivatives for θ and φ by the same manner. This gives for the three vectors e_1, e_2 and e_3:

$$e_1 = \partial_1 \mathbf{M} = sin\theta\ cos\varphi\ e_1^0 + sin\theta\ sin\varphi\ e_2^0 + cos\theta\ e_3^0 \qquad (50)$$

$$e_2 = \partial_2 \mathbf{M} = r\ cos\theta\ cos\varphi\ e_1^0 + r\ cos\theta\ sin\varphi\ e_2^0 - r\ sin\theta\ e_3^0 \qquad (51)$$

$$e_3 = \partial_3 \mathbf{M} = -r\ sin\theta\ sin\varphi\ e_1^0 + r\ sin\theta\ cos\varphi\ e_2^0 \qquad (52)$$

The vectors e_1^0, e_2^0 and e_3^0 are constant in module and direction. Therefore the differential of vectors e_1, e_2 and e_3 are:

$$\begin{aligned} de_1 = (&cos\theta\ cos\varphi\ e_1^0 + cos\theta\ sin\varphi\ e_2^0 - sin\theta\ e_3^0) d\theta \dots \\ &\dots + (-sin\theta\ sin\varphi\ e_1^0 + sin\theta\ cos\varphi\ e_2^0) d\varphi \end{aligned} \qquad (53)$$

$$\begin{aligned} de_2 = (&-r\ sin\theta\ cos\varphi\ e_1^0 - r\ sin\theta\ sin\varphi\ e_2^0 - r\ cos\theta\ e_3^0)d\theta \ldots \\ &\cdots + (-r\ cos\theta\ sin\varphi\ e_1^0 + r\ cos\theta\ cos\varphi\ e_2^0)d\varphi \ldots \\ &\cdots + (cos\theta\ cos\varphi\ e_1^0 + cos\theta\ sin\varphi\ e_2^0 - sin\theta\ e_3^0)dr \end{aligned} \tag{54}$$

$$\begin{aligned} de_3 = (&-r\ cos\theta\ sin\varphi\ e_1^0 + r\ cos\theta\ cos\varphi\ e_2^0)d\theta \ldots \\ &\cdots + (-r\ sin\theta\ cos\varphi\ e_1^0 - r\ sin\theta\ sin\varphi\ e_2^0)d\varphi \ldots \\ &\cdots + (-sin\theta\ sin\varphi\ e_1^0 + sin\theta\ cos\varphi\ e_2^0)dr \end{aligned} \tag{55}$$

We can remark that the terms in parenthesis are nothing but vectors e_1/r, e_2/r and e_3/r. This gives, after simplifications:

$$de_1 = (d\theta/r)e_2 + (d\varphi/r)e_3 \tag{56}$$

$$de_2 = (-r\ d\theta)e_1 + (dr/r)e_2 + (cotang\theta\ d\varphi)e_3 \tag{57}$$

$$de_3 = (-r\ sin^2\theta\ d\varphi)e_1 + (-sin\theta\ cos\theta\ d\varphi)e_2 + ((dr/r) + cotang\theta\ d\theta)e_3 \tag{58}$$

In a general manner, we can simplify the writing of this set of equation writing ω_i^j the contravariant components vectors de_i. The development of each term is given in the next section. The general expression, in 3D or more, is:

$$de_i = \omega_i^j e_j \tag{59}$$

Christoffel Symbols of the second kind

If we replace the variables r, θ and φ by u^1, u^2, and u^3 as follows:

$$u^1 = r; \quad u^2 = \theta; \quad u^3 = \varphi \tag{60}$$

...the differentials of the coordinates are:

$$du^1 = dr; \quad du^2 = d\theta; \quad du^3 = d\varphi \tag{61}$$

... and the ω_i^j components become, using the Christoffel symbol Γ_{ki}^j:

$$\omega_i^j = \Gamma_{ki}^j\ du^k \tag{62}$$

In the case of our example, quantities Γ^j_{ki} are functions of r, θ and φ. These functions can be explicitly obtained by an identification of each component of ω^j_i with Γ^j_{ki}. The full development of the precedent expressions of our example is detailed as follows:

$$\begin{cases} \omega^1_1 = 0 \\ \omega^2_1 = 1/r \ d\theta \\ \omega^3_1 = 1/r \ d\varphi \\ \omega^1_2 = -r \ d\theta \\ \omega^2_2 = 1/r \ dr \\ \omega^3_2 = cotang \ \theta \ d\varphi \\ \omega^1_3 = -r \ sin^2\theta \ d\theta \\ \omega^2_3 = -sin\theta \ cos\theta \ d\varphi \\ \omega^3_3 = 1/r \ dr + cotang\theta \ d\theta \end{cases} \tag{63}$$

Replacing dr by du^1, $d\theta$ by du^2, and $d\varphi$ by du^3 as indicated in equation (61), gives:

$$\begin{cases} \omega^1_1 = 0 \\ \omega^2_1 = 1/r \ du^2 \\ \omega^3_1 = 1/r \ du^3 \\ \omega^1_2 = -r \ du^2 \\ \omega^2_2 = 1/r \ du^1 \\ \omega^3_2 = cotang \ \theta \ du^3 \\ \omega^1_3 = -r \ sin^2\theta \ du^2 \\ \omega^2_3 = -sin\theta \ cos\theta \ du^3 \\ \omega^3_3 = 1/r \ du^1 + cotang \ \theta \ du^2 \end{cases} \tag{64}$$

On the other hand, the development of Christoffel symbols are:

$$\begin{cases} \omega^1_1 = \Gamma^1_{11} \ du^1 + \Gamma^1_{21} \ du^2 + \Gamma^1_{31} \ du^3 \\ \omega^2_1 = \Gamma^2_{11} \ du^1 + \Gamma^2_{21} \ du^2 + \Gamma^2_{31} \ du^3 \\ \omega^3_1 = \Gamma^3_{11} \ du^1 + \Gamma^3_{21} \ du^2 + \Gamma^3_{31} \ du^3 \\ \omega^1_2 = \Gamma^1_{12} \ du^1 + \Gamma^1_{22} \ du^2 + \Gamma^1_{32} \ du^3 \\ \omega^2_2 = \Gamma^2_{12} \ du^1 + \Gamma^2_{22} \ du^2 + \Gamma^2_{32} \ du^3 \\ \omega^3_2 = \Gamma^3_{12} \ du^1 + \Gamma^3_{22} \ du^2 + \Gamma^3_{32} \ du^3 \\ \omega^1_3 = \Gamma^1_{13} \ du^1 + \Gamma^1_{23} \ du^2 + \Gamma^1_{33} \ du^3 \\ \omega^2_3 = \Gamma^2_{13} \ du^1 + \Gamma^2_{23} \ du^2 + \Gamma^2_{33} \ du^3 \\ \omega^3_3 = \Gamma^3_{13} \ du^1 + \Gamma^3_{23} \ du^2 + \Gamma^3_{33} \ du^3 \end{cases} \tag{65}$$

Finally, identifying the two equations array (64) and (65) gives the 27 Christoffel Symbols.

$$
\begin{array}{l|l|l}
\Gamma^1_{11} = 0 & \Gamma^1_{21} = 0 & \Gamma^1_{31} = 0 \\
\Gamma^2_{11} = 0 & \Gamma^2_{21} = 1/r & \Gamma^2_{31} = 0 \\
\Gamma^3_{11} = 0 & \Gamma^3_{21} = 0 & \Gamma^3_{31} = 1/r \\
\Gamma^1_{12} = 0 & \Gamma^1_{22} = -r & \Gamma^1_{32} = 0 \\
\Gamma^2_{12} = 0 & \Gamma^2_{22} = 0 & \Gamma^2_{32} = 0 \\
\Gamma^3_{12} = 0 & \Gamma^3_{22} = 0 & \Gamma^3_{32} = cotang\ \theta \\
\Gamma^1_{13} = 0 & \Gamma^1_{23} = -r\ sin^2\theta & \Gamma^1_{33} = 0 \\
\Gamma^2_{13} = 0 & \Gamma^2_{23} = 0 & \Gamma^2_{33} = -sin\theta\ cos\theta \\
\Gamma^3_{13} = 1/r & \Gamma^3_{23} = cotang\ \theta & \Gamma^3_{33} = 0
\end{array}
\qquad (66)
$$

These quantities Γ^j_{ki} are the Christoffel Symbols of the second kind. Identifying equations (59) and (62) gives the general expression of the Christoffel Symbols of the second kind:

$$\boxed{de_i = \omega^j_i e_j = \Gamma^j_{ki}\ du^k\ e_j} \qquad (67)$$

Christoffel Symbols of the first kind

We have seen in the precedent example that we can directly get the quantities Γ^j_{ki} by identification. These quantities can also be obtained from the components g_{ij} of the metric tensor. This calculus leads to another kind of Christoffel Symbols.

Lets write the covariant components, noted ω_{ji}, of the differentials de_i:

$$\omega_{ji} = e_j\ de_i \qquad (68)$$

The covariant components ω_{ji} are also linear combinations of differentials du^i that can be written as follows, using the Christoffel Symbol of the first kind Γ_{kji} :

$$\omega_{ji} = \Gamma_{kji}\ du^k \qquad (69)$$

On the other hand, we know the basic relation:

$$\omega_{ji} = g_{jl}\ \omega^l_i \qquad (70)$$

Porting equation (69) in equation (70) gives:

$$\Gamma_{kji}\ du^k = g_{jl}\ \omega^l_i \qquad (71)$$

Let's change the name of index j to l of equation (62):

$$\omega_i^l = \Gamma_{ki}^l \ du^k \qquad (72)$$

Porting equation (117) in equation (116) gives the calculus of the Christoffel Symbols of the first kind from the Christoffel Symbols of the second kind:

$$\boxed{\Gamma_{kji} \ du^k = g_{jl} \ \Gamma_{ki}^j \ du^k} \qquad (73)$$

Geodesic Equations

Let's take a curve M_0-C-M_1. If the parametric equations of the curvilinear abscissa are $u^i(s)$, the length of the curve will be:

$$l = \int_{M_0}^{M_1} \left(g_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \right)^{1/2} ds \qquad (74)$$

If we pose $u'^i = du^i/ds$ and $u'^j = du^j/ds$ we get:

$$l = \int_{M_0}^{M_1} \left(g_{ij} u'^i u'^j \right)^{1/2} ds \qquad (75)$$

Here, the u'^j are the direction cosines of the unit vector supported by the tangent to the curve. Thus, we can pose:

$$f(u^k, u'^j) = g_{ij} u'^i u'^j = 1 \qquad (76)$$

The length l of the curve defined by equation (74) has a minimum and a maximum that can be calculated by the Euler-Lagrange Equation which is:

$$\frac{\partial \mathcal{L}}{\partial f_i} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial f_i'} \right) = 0 \qquad (77)$$

In the case of equation (74), the Euler-Lagrange equation gives:

$$\frac{d}{ds}(g_{ij} u'^j) - \frac{1}{2} \partial_i g_{jk} \ u'^j \ u'^k = 0 \qquad (78)$$

or

$$g_{ij}u'^j + (\partial_k g_{ij} - \frac{1}{2}\partial_i g_{jk})\ u'^j\ u'^k = 0 \qquad (79)$$

After developing derivative and using the Christoffel Symbol of the first kind, we get:

$$g_{ij}\frac{du'^j}{ds} + \Gamma_{jik}\ u'^j\ u'^k = 0 \qquad (80)$$

The contracted multiplication of equation (79) by g^{il}gives, with $g_{ij}g^{il}$ and $g^{il}\Gamma_{ijk} = \Gamma^l_{jk}$:

$$\frac{d^2u^l}{ds^2} + \Gamma^l_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0 \qquad (81)$$

Parallel Transport

Figure 2 shows two points M and M' infinitely close to each other in polar coordinates.

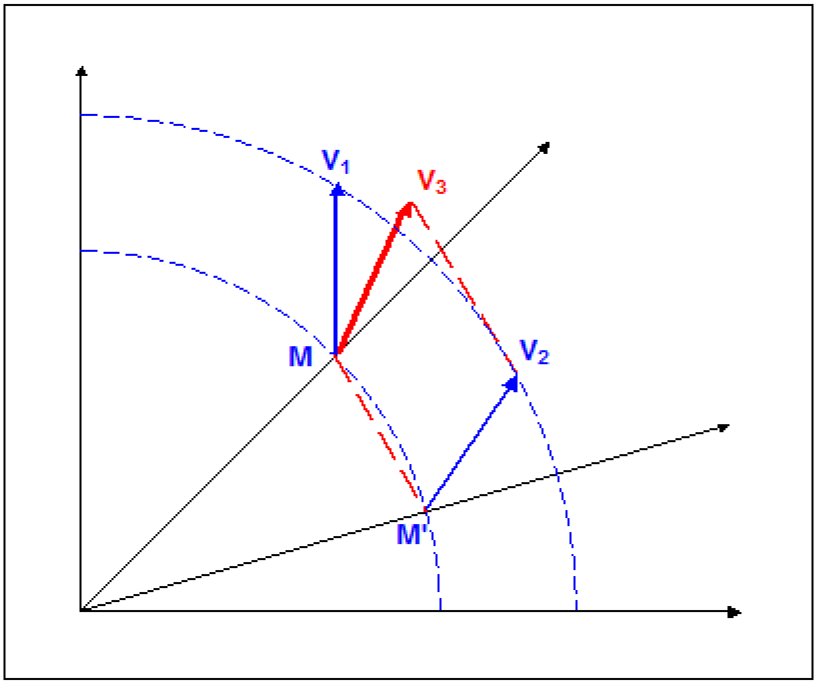

Figure 2: Parallel transport of a vector.

In polar coordinates, the vector V_1 will become V_2. To calculate the difference between two vectors V_1 and V_2, we must before make a parallel transport of the vector V_2 from point M' to point M. This gives the vector V_3. The absolute differential is defined by:

$$dV = V_3 - V_1 \qquad (82)$$

Variations along a Geodesic

The variation along a 4D geodesic follows the same principle. For any curvilinear coordinates system y^i, we have (from equation 81):

$$\frac{d^2y^i}{ds^2} + \Gamma^i_{kj}\frac{dy^j}{ds}\frac{dy^k}{ds} = 0 \qquad (83)$$

where s is the abscissa of any point of the straight line from an origin such as M in figure 2.

Let's consider now a vector $\vec{v}$ having covariant components v_i. We can calculate the scalar product of $\vec{v}$ and $\vec{n} = dy^k/ds$ as follows:

$$\vec{v}\cdot\vec{n} = v_i\frac{dy^i}{ds} \qquad (84)$$

During a displacement from M to M' (figure 2), the scalar is subjected to a variation of:

$$d\left(v_i\frac{dy^i}{ds}\right) = dv_k\frac{dy^k}{ds} + v_i\ d\left(\frac{d^2y^i}{ds^2}\right) \qquad (85)$$

or:

$$d\left(v_i\frac{dy^i}{ds}\right) = dv_k\frac{dy^k}{ds} + v_i\frac{d^2y^i}{ds^2}ds \qquad (86)$$

On one hand, the differential dv_k can be written as:

$$dv_k = \partial_j v_k\frac{dy^j}{ds}ds \qquad (87)$$

On the other hand, the second derivative can be extracted from equation (83) as follows:

$$\frac{d^2y^i}{ds^2} = -\Gamma^i_{kj}\frac{dy^j}{ds}\frac{dy^k}{ds} \qquad (88)$$

Porting equations (87) and (88) in (86) gives:

$$d\left(v_i \frac{dy^i}{ds}\right) = \partial_j v_k \frac{dy^j}{ds}\frac{dy^k}{ds} ds - v_i \Gamma^i_{kj} \frac{dy^j}{ds}\frac{dy^k}{ds} ds \qquad (89)$$

or:

$$d\left(v_i \frac{dy^i}{ds}\right) = \left(\partial_j v_k - v_i \Gamma^i_{kj}\right) \frac{dy^j}{ds}\frac{dy^k}{ds} ds \qquad (90)$$

Since $(dy^j/ds)ds = dy^j$, this expression can also be written as follows:

$$d(\vec{v} \cdot \vec{n}) = \left(\partial_j v_k - v_i \Gamma^i_{kj}\right) dy^j \frac{dy^k}{ds} \qquad (91)$$

The absolute differentials of the covariant components of vector $\vec{v}$ are defined as:

$$D v_k \frac{dy^k}{ds} = \left(\partial_j v_k - v_i \Gamma^i_{kj}\right) dy^j \qquad (92)$$

Finally, the quantity in parenthesis is called "affine connection" and is defined as follows:

$$\boxed{\nabla_j v_k = \partial_j v_k - v_i \Gamma^i_{kj}} \qquad (93)$$

Some countries in the world use ";" for the covariant derivative and "," for the partial derivative. Using this convention, equation (93) can be written as:

$$v_{k;j} = v_{k,j} - v_i \Gamma^i_{kj} \qquad (94)$$

To summarize, given a function f, the covariant derivative $\nabla_v f$ coincides with the normal differentiation of a real function in the direction of the vector $\vec{v}$, usually denoted by $\vec{v}f$ and $df(\vec{v})$.

Second Covariant Derivatives of a Vector

Remembering that the derivative of the product of two functions is the sum of partial derivatives, we have:

$$\nabla_a(t_b r_c) = r_c \; . \; \nabla_a t_b + t_b \; . \; \nabla_a r_c \qquad (95)$$

Porting equation (93) in equation (95) gives:

$$\nabla_a(t_b r_c) = r_c \left(\partial_a t_b - t_l \Gamma^l_{ab}\right) + t_b \left(\partial_a r_c - r_l \Gamma^l_{ac}\right) \qquad (96)$$

or:

$$\nabla_a(t_b r_c) = r_c\partial_a t_b - r_c t_l \Gamma^l_{ab} + t_b\partial_a r_c - t_b r_l \Gamma^l_{ac} \qquad (97)$$

Hence:

$$\nabla_a(t_b r_c) = r_c\partial_a t_b + t_b\partial_a r_c - r_c t_l \Gamma^l_{ab} - t_b r_l \Gamma^l_{ac} \qquad (98)$$

Finally:

$$\nabla_a(t_b r_c) = \partial_a(t_b r_c) - r_c t_l \Gamma^l_{ab} - t_b r_l \Gamma^l_{ac} \qquad (99)$$

Posing $t_b r_c = \nabla_j v_i$ gives:

$$\boxed{\nabla_k(\nabla_j v_i) = \partial_k(\nabla_j v_i) - (\nabla_j v_r)\Gamma^r_{ik} - (\nabla_r v_i)\Gamma^r_{jk}} \qquad (100)$$

Porting equation (93) in equation (100) gives:

$$\nabla_k(\nabla_j v_i) = \partial_k(\partial_j v_i - v_l\Gamma^l_{ji}) - (\partial_j v_r - v_l\Gamma^l_{jr})\Gamma^r_{ik} - (\partial_r v_i - v_l\Gamma^l_{ri})\Gamma^r_{jk} \qquad (101)$$

Hence:

$$\nabla_k(\nabla_j v_i) = \partial_{kj} v_i - (\partial_k\Gamma^l_{ji})v_l - \Gamma^l_{ji}\partial_k v_l - \Gamma^r_{ik}\partial_j v_r + \Gamma^r_{ik}\Gamma^l_{jr} v_l - \Gamma^r_{jk}\partial_r v_i + \Gamma^r_{jk}\Gamma^l_{ri} v_l \qquad (102)$$

The Riemann-Christoffel Tensor

In expression (102), if we make a swapping between the indexes j and k in order to get a differential on another way (i.e. a parallel transport) we get:

$$\nabla_j(\nabla_k v_i) = \partial_{jk} v_i - (\partial_j\Gamma^l_{ki})v_l - \Gamma^l_{ik}\partial_j v_l - \Gamma^r_{ij}\partial_k v_r + \Gamma^r_{ij}\Gamma^l_{kr} v_l - \Gamma^r_{kj}\partial_r v_i + \Gamma^r_{kj}\Gamma^l_{ri} v_l \qquad (103)$$

A subtraction between expressions (102) and (103) gives, after a rearrangement of some terms:

$$\nabla_k(\nabla_j v_i) - \nabla_j(\nabla_k v_i) = (\partial_{kj} - \partial_{jk})v_i + (\partial_j\Gamma^l_{ki} - \partial_k\Gamma^l_{ji})v_l + (\Gamma^l_{ik}\partial_j - \Gamma^l_{ji}\partial_k)v_l \ldots$$
$$\cdots + (\Gamma^r_{ij}\partial_k - \Gamma^r_{ik}\partial_j)v_r + (\Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr})v_l + (\Gamma^r_{kj} - \Gamma^r_{jk})\partial_r v_i + (\Gamma^r_{jk}\Gamma^l_{ri} - \Gamma^r_{kj}\Gamma^l_{ri})v_l \quad (104)$$

On the other hand, since we have:

$$\Gamma^r_{jk} = \Gamma^r_{kj} \qquad (105)$$

Some terms of equation (104) are canceled:

$$\partial_{kj} - \partial_{jk} = 0 \qquad (106)$$

$$\Gamma^r_{kj} - \Gamma^r_{jk} = 0 \qquad (107)$$

$$\Gamma^r_{jk}\Gamma^l_{ri} - \Gamma^r_{kj}\Gamma^l_{ri} = 0 \qquad (108)$$

And consequently:

$$\nabla_k(\nabla_j v_i) - \nabla_j(\nabla_k v_i) = (\partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji})v_l + (\Gamma^l_{ik}\partial_j - \Gamma^l_{ji}\partial_k)v_l \ldots$$
$$\cdots + (\Gamma^r_{ij}\partial_k - \Gamma^r_{ik}\partial_j)v_r + (\Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr})v_l \quad (109)$$

Since the parallel transport is done on small portions of geodesics infinitely close to each other, we can take the limit:

$$\partial_j v_l, \quad \partial_k v_l, \quad \partial_j v_r, \quad \partial_k v_r \;\rightarrow\; 0 \qquad (110)$$

This means that the velocity field is considered equal in two points of two geodesics infinitely close to each other. Then we can write:

$$\nabla_k(\nabla_j v_i) - \nabla_j(\nabla_k v_i) \cong (\partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr})v_l \quad (111)$$

As a result of the tensorial properties of covariant derivatives and of the components v_l, the quantity in parenthesis is a four-order tensor defined as:

$$\boxed{R^l_{i,jk} = \partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{ij}\Gamma^l_{kr}} \qquad (112)$$

In this expression, the comma in Christoffel Symbols means a partial derivative. The tensor $R^l_{i,jk}$ is called *Riemann-Christoffel Tensor* or *Curvature Tensor* which characterizes the curvature of a Riemann Space.

The Ricci Tensor

The contraction of the Riemann-Christoffel Tensor $R^l_{i,jk}$ defined by equation (112) relative to indexes l and j leads to a new tensor:

$$\boxed{R_{ik} = R^l_{i,lk} = \partial_l \Gamma^l_{ki} - \partial_k \Gamma^l_{li} + \Gamma^r_{ik}\Gamma^l_{lr} - \Gamma^r_{il}\Gamma^l_{kr}} \qquad (113)$$

This tensor R_{ik} is called the *"Ricci Tensor"*. Its mixed components are given by:

$$R^{jk} = g^{ji} R_{ik} \quad (114)$$

The Scalar Curvature

The *Scalar Curvature*, also called the *"Curvature Scalar"* or *"Ricci Scalar"*, is given by:

$$\boxed{R = R^i_i = g^{ij} R_{ij}} \qquad (115)$$

The Bianchi Second Identities

The Riemann-Christoffel tensor verifies a particular differential identity called the *"Bianchi Identity"*. This identity involves that the Einstein Tensor has a null divergence, which leads to a constraint. The goal is to reduce the degrees of freedom of the Einstein Equations. To calculate the second Bianchi Identities, we must derivate the Riemann-Christoffel Tensor defined in equation (112):

$$\nabla_t R^l_{i,rs} = \partial_{rt} \Gamma^l_{si} - \partial_{st} \Gamma^l_{ri} \qquad (116)$$

A circular permutation of indexes r, s and t gives:

$$\nabla_r R^l_{i,st} = \partial_{sr} \Gamma^l_{ti} - \partial_{tr} \Gamma^l_{si} \qquad (117)$$

$$\nabla_s R^l_{i,tr} = \partial_{ts} \Gamma^l_{ri} - \partial_{rs} \Gamma^l_{ti} \qquad (118)$$

Since the derivation order is interchangeable, adding equations (116), (117) and (118) gives:

$$\nabla_t R^l_{i,rs} + \nabla_r R^l_{i,st} + \nabla_s R^l_{i,tr} = 0 \qquad (119)$$

The Einstein Tensor

If we make a contraction of the second Bianchi Identities (equation 119) for $t = l$, we get:

$$\nabla_l R^l_{i,rs} + \nabla_r R^l_{i,sl} + \nabla_s R^l_{i,lr} = 0 \qquad (120)$$

Hence, taking into account and the definition of the Ricci Tensor of equation (113) and that $R^l_{i,sl} = -R^l_{i,ls}$, we get:

$$\nabla_l R^l_{i,rs} + \nabla_s R_{ir} - \nabla_r R_{is} = 0 \qquad (121)$$

The variance change with g_{ij} gives:

$$\nabla_s R_{ir} = \nabla_s (g_{ik} R^k_r) \qquad (122)$$

or

$$\nabla_s R_{ir} = g_{ik} \nabla_s R^k_r \qquad (123)$$

Multiplying equation (121) by g^{ik} gives:

$$g^{ik} \nabla_l R^l_{i,rs} + g^{ik} \nabla_s R_{ir} - g^{ik} \nabla_r R_{is} = 0 \qquad (124)$$

Using the property of equation (123), we finally get:

$$\nabla_l R^{kl}_{,rs} + \nabla_s R^k_r - \nabla_r R^k_s = 0 \qquad (125)$$

Let's make a contraction on indexes k and s:

$$\nabla_k R^{kl}_{,rk} + \nabla_k R^k_r - \nabla_r R^k_k = 0 \qquad (126)$$

The first term becomes:

$$\nabla_k R^k_r + \nabla_k R^k_r - \nabla_r R^k_k = 0 \qquad (127)$$

After a contraction of the third term we get:

$$2\nabla_k R^k_r - \nabla_r R = 0 \qquad (128)$$

Dividing this expression by two gives:

$$\nabla_k R^k_r - \frac{1}{2} \nabla_r R = 0 \qquad (129)$$

or:

$$\nabla_k \left(R_r^k - \frac{1}{2}\delta_r^k R \right) = 0 \qquad (130)$$

A new tensor may be written as follows:

$$G_r^k = R_r^k - \frac{1}{2}\delta_r^k R \qquad (131)$$

The covariant components of this tensor are:

$$G_{ij} = g_{ik}\ G_j^k \qquad (132)$$

or

$$G_{ij} = g_{ik} \left(R_j^k - \frac{1}{2}\delta_j^k R \right) \qquad (133)$$

Finally get the Einstein Tensor which is defined by:

$$\boxed{G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R} \qquad (134)$$

The Einstein Constant

The Einstein Tensor G_{ij} of equation (134) must match the Energy-Momentum Tensor T_{uv} defined later. This can be done with a constant κ so that:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad (135)$$

This constant κ is called *"Einstein Constant"* or *"Constant of Proportionality"*. To calculate it, the Einstein Equation (134) must be identified to the Poisson's classical field equation, which is the mathematical form of the Newton Law. So, the weak field approximation is used to calculate the Einstein Constant. Three criteria are used to get this "Newtonian Limit":

1 - The speed is low regarding that of the light c.
2 - The gravitational field is static.
3 - The gravitational field is weak and can be seen as a weak perturbation $h_{\mu\nu}$ added to a flat spacetime $\eta_{\mu\nu}$ as follows:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (136)$$

We start with the equation of geodesics (83):

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad (137)$$

This equation can be simplified in accordance with the first condition:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{ds}\right)^2 = 0 \qquad (138)$$

The two other conditions lead to a simplification of Christoffel Symbols of the second kind as follows:

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g00) \qquad (139)$$

Or, considering the second condition:

$$\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda\ g_{00} \qquad (140)$$

And also considering the third condition:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}(\eta^{\mu\lambda} + h^{\mu\lambda})(\partial_\lambda\eta_{00} + \partial_\lambda h_{00}) \qquad (141)$$

In accordance with the third condition, the term $\partial_\lambda\eta_{00}$ is canceled since it is a flat space:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}(\eta^{\mu\lambda} + h^{\mu\lambda})\partial_\lambda h_{00} \qquad (142)$$

Another simplification due to the approximation gives:

$$\Gamma^\mu_{00} \approx -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \qquad (143)$$

The equation of geodesics then becomes:

$$\frac{d^2x^\mu}{dt^2} - \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad (144)$$

Reduced to the time component ($\mu = 0$), equation (144) becomes:

$$\frac{d^2x^\mu}{dt^2} - \frac{1}{2}\eta^{0\lambda}\partial_\lambda h_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad (145)$$

The Minkowski Metric shows that $\eta_{0\lambda} = 0$ for $\lambda > 0$. On the other hand, a static metric (third condition) gives $\partial_0 h_{00} = 0$ for $\lambda = 0$. So, the 3x3 matrix leads to:

$$\frac{d^2x^i}{dt^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{dt}\right)^2 = 0 \qquad (146)$$

Replacing dt by the proper time $d\tau$ gives:

$$\frac{d^2x^i}{d\tau^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0 \qquad (147)$$

Dividing by $(dx_0/d\tau)^2$ leads to:

$$\frac{d^2x^i}{d\tau^2}\left(\frac{d\tau}{dx^0}\right)^2 = \frac{1}{2}\partial_i h_{00} \qquad (148)$$

$$\frac{d^2x^i}{(dx^0)^2} = \frac{1}{2}\partial_i h_{00} \qquad (149)$$

Replacing x^0 by ct gives:

$$\frac{d^2x^i}{d(ct)^2} = \frac{1}{2}\partial_i h_{00} \qquad (150)$$

or:

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\partial_i h_{00} \qquad (151)$$

Let us pose:

$$h_{00} = -\frac{2}{c^2}\Phi \qquad (152)$$

or

$$\triangle h_{00} = -\frac{2}{c^2}\triangle\Phi \qquad (153)$$

Since the approximation is in an euclidean space, the Laplace operator can be written as:

$$\triangle h_{00} = -\frac{2}{c^2}\nabla^2\Phi \qquad (154)$$

$$\triangle h_{00} = -\frac{2}{c^2}\left(4\pi G_0\rho\right) \qquad (155)$$

$$\triangle h_{00} = -\frac{8\pi G_0}{c^2}\rho \qquad (156)$$

On the other hand, the element T_{00} defined later is:

$$T_{00} = \rho c^2 \qquad (157)$$

or

$$\rho = \frac{T_{00}}{c^2} \qquad (158)$$

Porting equation (158) in equation (156) gives:

$$\triangle h_{00} = -\frac{8\pi G_0}{c^2}\frac{T_{00}}{c^2} \qquad (159)$$

or:

$$\triangle h_{00} = -\frac{8\pi G_0}{c^4}T_{00} \qquad (160)$$

The left hand side of equation (160) is the perturbation part of the Einstein Tensor in the case of a static and weak field approximation. It directly gives a constant of proportionality which also verifies the homogeneity of the EFE (equation 1). This equation will be fully explained later in this document. Thus:

$$\boxed{Einstein\ Constant = \frac{8\pi G_0}{c^4}} \qquad (161)$$

4 - Movement Equations in a Newtonian Fluid

The figure 3 (next page) shows an elementary parallelepiped of dimensions dx, dy, dz which is a part of a fluid in static equilibrium. This cube is generally subject to volume forces in all directions, as the Pascal Theorem states. The components of these forces are oriented in the three orthogonal axis. The six sides of the cube are: A-A', B-B' and C-C'.

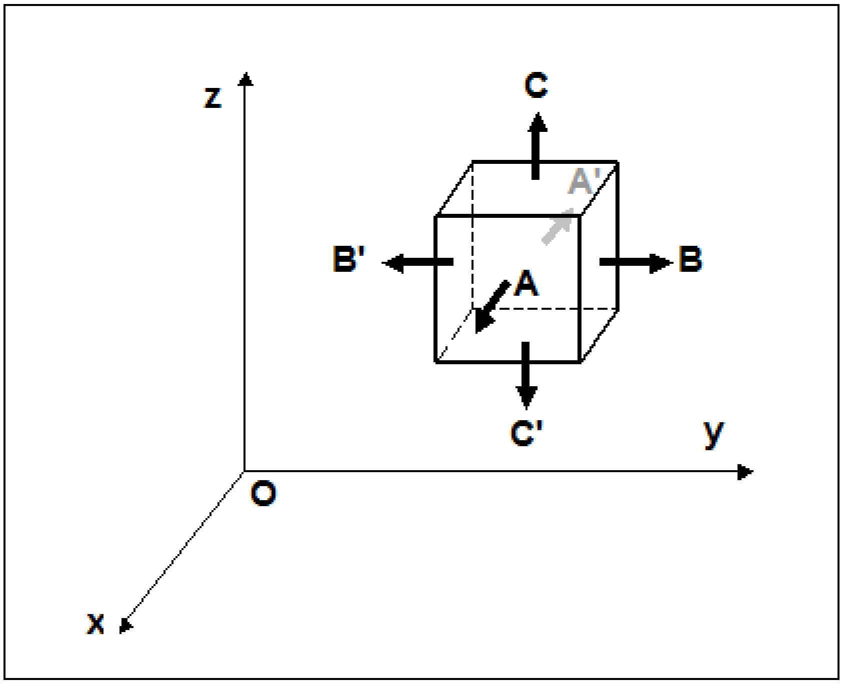

Figure 3: Elementary parallelepiped dx, dy, dz

5 - Normal Constraints

On figure 4 (next page), the normal constraints to each surface are noted "σ". The tangential constraints to each surface are noted "τ". Since we have six sides, we have six sets of equations. In the following equations, 'σ" and "τ' are constraints (a constraint is a pressure), dF is an elementary force, and dS is an elementary surface:

$$\frac{d\vec{F}_{+x}}{dS} = \vec{\sigma}_x + \vec{\tau}'_{yx} + \vec{\tau}_{xz} \qquad (162)$$

$$\frac{d\vec{F}_{-x}}{dS} = \vec{\sigma}'_x + \vec{\tau}'_{xy} + \vec{\tau}_{zx} \qquad (163)$$

$$\frac{d\vec{F}_{+y}}{dS} = \vec{\sigma}_y + \vec{\tau}_{yz} + \vec{\tau}''_{yx} \qquad (164)$$

$$\frac{d\vec{F}_{-y}}{dS} = \vec{\sigma}'_y + \vec{\tau}_{zy} + \vec{\tau}''_{xy} \qquad (165)$$

$$\frac{d\vec{F}_{+z}}{dS} = \vec{\sigma}_z + \vec{\tau}''_{xz} + \vec{\tau}'_{yz} \qquad (166)$$

$$\frac{d\vec{F}_{-z}}{dS} = \vec{\sigma}'_z + \vec{\tau}''_{zx} + \vec{\tau}'_{zy} \qquad (167)$$

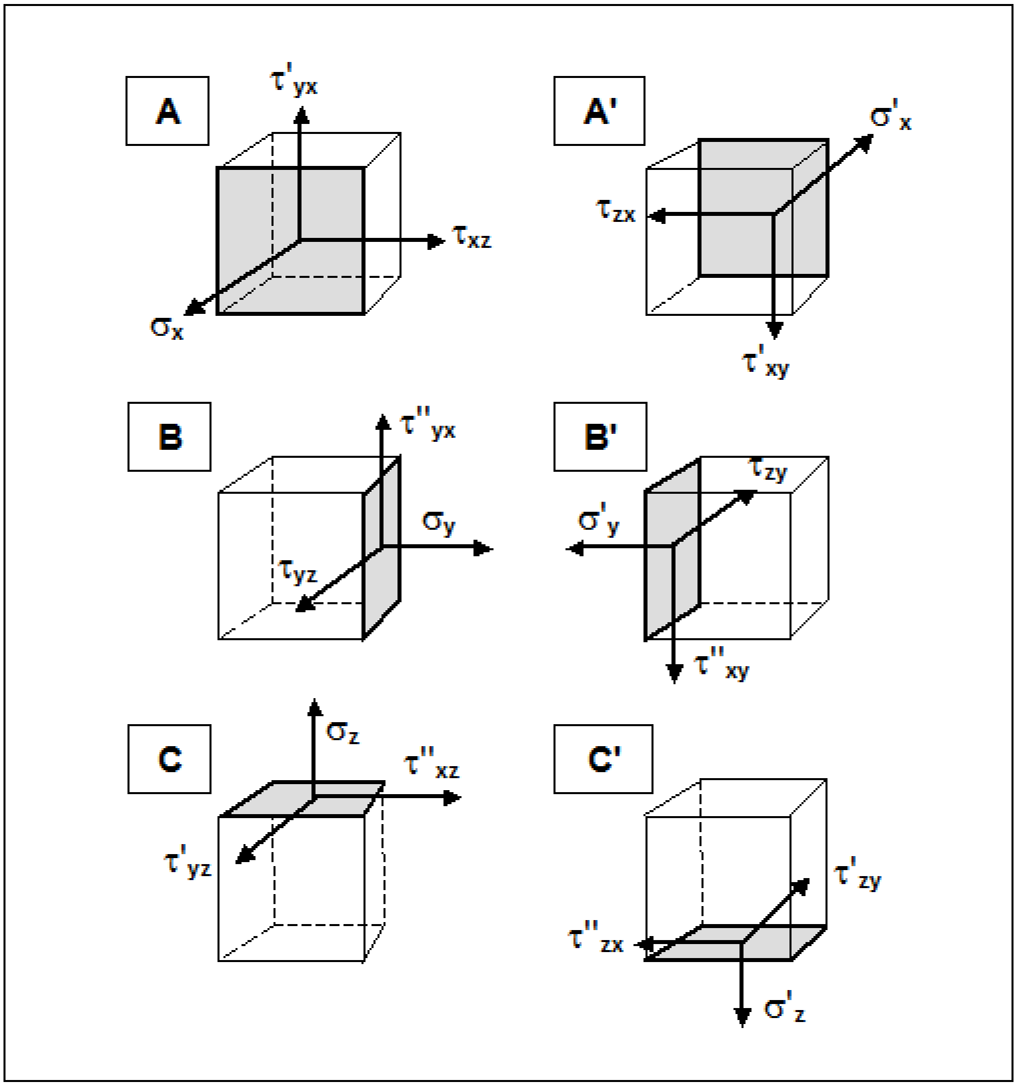

Figure 4: Forces on the elementary parallelepiped sides.

We can simplify these equations as follows:

$$\vec{\sigma}_x + \vec{\sigma}'_x = \vec{\sigma}_X \qquad (168)$$

$$\vec{\sigma}_y + \vec{\sigma}'_y = \vec{\sigma}_Y \qquad (169)$$

$$\vec{\sigma}_z + \vec{\sigma}'_z = \vec{\sigma}_Z \qquad (170)$$

So, only three components are used to define the normal constraint forces, i.e. one per axis.

6 - Tangential Constraints

If we calculate the force's momentum regarding the gravity center of the parallelepiped, we have 12 tangential components (two per side). Since some forces are in opposition to each other, only 6 are sufficient to describe the system. Here, we calculate the three momenta for each plan, XOY, XOZ and YOZ, passing through the gravity center of the elementary parallelepiped:

For the XOY plan:

$$\mathcal{M}_{XOY} = (\vec{\tau}_{zy} dzdx)\frac{dy}{2} + (\vec{\tau}_{zx} dydz)\frac{dx}{2} + (\vec{\tau}_{yz} dxdz)\frac{dy}{2} + (\vec{\tau}_{xz} dzdy)\frac{dx}{2} \quad (171)$$

$$\mathcal{M}_{XOY} = \frac{1}{2} dV \left[(\vec{\tau}_{zy} + \vec{\tau}_{yz}) + (\vec{\tau}_{zx} + \vec{\tau}_{xz})\right] \quad (172)$$

$$\mathcal{M}_{XOY} = \frac{1}{2} dV \left[(\vec{\tau}_{ZY} + \vec{\tau}_{ZX})\right] \quad (173)$$

$$\mathcal{M}_{XOY} = \frac{1}{2} dV \vec{\tau}_{XOY} \quad (174)$$

For the XOZ plan:

$$\mathcal{M}_{XOZ} = (\vec{\tau}'_{yx} dydz)\frac{dx}{2} + (\vec{\tau}'_{yz} dxdy)\frac{dz}{2} + (\vec{\tau}'_{xy} dzdy)\frac{dx}{2} + (\vec{\tau}'_{zy} dydx)\frac{dz}{2} \quad (175)$$

$$\mathcal{M}_{XOZ} = \frac{1}{2} dV \left[(\vec{\tau}'_{yx} + \vec{\tau}'_{xy}) + (\vec{\tau}'_{yz} + \vec{\tau}'_{zy})\right] \quad (176)$$

$$\mathcal{M}_{XOZ} = \frac{1}{2} dV \left[(\vec{\tau}_{YX} + \vec{\tau}_{YZ})\right] \quad (177)$$

$$\mathcal{M}_{XOZ} = \frac{1}{2} dV \vec{\tau}_{XOZ} \quad (178)$$

For the ZOY plan:

$$\mathcal{M}_{ZOY} = (\vec{\tau}''_{xz} dxdy)\frac{dz}{2} + (\vec{\tau}''_{yx} dxdz)\frac{dy}{2} + (\vec{\tau}''_{zx} dydx)\frac{dz}{2} + (\vec{\tau}''_{xy} dzdx)\frac{dy}{2} \quad (179)$$

$$\mathcal{M}_{ZOY} = \frac{1}{2} dV \left[(\vec{\tau}''_{xz} + \vec{\tau}''_{zx}) + (\vec{\tau}''_{yx} + \vec{\tau}''_{xy})\right] \quad (180)$$

$$\mathcal{M}_{ZOY} = \frac{1}{2} dV \left[(\vec{\tau}_{ZX} + \vec{\tau}_{XY})\right] \quad (181)$$

$$\mathcal{M}_{ZOY} = \frac{1}{2} dV \vec{\tau}_{ZOY} \quad (182)$$

So, for each plan, only one component is necessary to define the set of momentum forces. Since the elementary volume dV is equal to a*a*a (a = dx, dy or dz), we can come back to the constraint equations dividing each result (174), (178) and (182) by dV/2.

7 - Constraint Tensor

Finally, the normal and tangential constraints can be reduced to only 6 terms with $\vec{\tau}_{xy} = \vec{\tau}_{yx}$, $\vec{\tau}_{xz} = \vec{\tau}_{zx}$, and $\vec{\tau}_{zy} = \vec{\tau}_{yz}$:

$$\vec{\sigma}_X = \vec{\sigma}_{xx} \tag{183}$$
$$\vec{\sigma}_Y = \vec{\sigma}_{yy} \tag{184}$$
$$\vec{\sigma}_Z = \vec{\sigma}_{zz} \tag{185}$$
$$\vec{\tau}_{XOY} = \vec{\tau}_{xy} \tag{186}$$
$$\vec{\tau}_{XOZ} = \vec{\tau}_{xz} \tag{187}$$
$$\vec{\tau}_{ZOY} = \vec{\tau}_{zy} \tag{188}$$

Using a matrix representation, we get:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} \tag{189}$$

The constraint tensor at point M becomes:

$$T_{(M)} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \tag{190}$$

This tensor is symmetric and its meaning is shown in Figure 5.

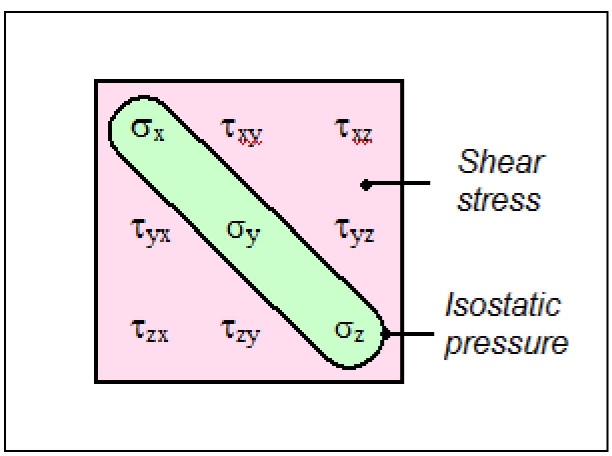

Figure 5: Meaning of the Constraint Tensor.

Since all the components of the tensor are pressures (more exactly constraints), we can represent it by the following equation where F_i are forces and s_i are surfaces:

$$\boxed{F_i = \sigma_{ij} s_j = \sum_j \sigma_{ij} s_j} \qquad (191)$$

8 - Energy-Momentum Tensor

We can write equation (191) as:

$$\sigma_{ij} = \frac{F_i}{s_j} = \frac{\Delta F_i}{\Delta s_j} = \frac{\Delta(ma_i)}{\Delta s_j} \qquad (192)$$

Here we suppose that only volumes (x, y and z) and time (t) make the force vary. Therefore, the mass (m) can be replaced by volume (V) using a constant density (ρ):

$$m = \rho V \qquad (193)$$

Porting (193) in (192) gives:

$$\sigma_{ij} = \frac{\Delta(m.a_i)}{\Delta s_j} = \frac{\Delta(\rho V.a_i)}{\Delta s_j} = \frac{\rho V}{\Delta s_j} \Delta a_i \qquad (194)$$

As shown in figure 3, the volume V and surface S_j (for surface on side j), concern an elementary parallelepiped.

$$V = (\Delta X_j)^3 \quad and \quad S_j = (\Delta X_j)^2 \qquad (195)$$

Thus:

$$\frac{V}{S_j} = \frac{(\Delta X_j)^3}{(\Delta X_j)^2} = \Delta X_j \qquad (196)$$

Hence

$$\sigma_{ij} = \rho \ \Delta X_j \Delta a_i \qquad (197)$$

Since Δa_i is an acceleration, or v_i/dt:

$$\sigma_{ij} = \rho \Delta X_j \frac{v_i}{\Delta t} = \rho \frac{\Delta X_j}{\Delta t} v_i \qquad (198)$$

Finally

$$\sigma_{ij} = \rho v_i v_j \qquad (199)$$

This tensor comes from the Fluid Mechanics and uses traditional variables v_x, v_y and v_z. We can extend these 3D variables to 4D in accordance with Special Relativity (see above). The new 4D tensor created, called the "Energy-Momentum Tensor', has the same properties as the old one, in particular symmetry. To avoid confusion, let's replace σ_{ij} by $T_{\mu\nu}$ as follows:

$$\boxed{T_{\mu\nu} = \rho u_\mu u_\nu} \tag{200}$$

or

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \rho u_0 u_0 & \rho u_0 u_1 & \rho u_0 u_2 & \rho u_0 u_3 \\ \rho u_1 u_0 & \rho u_1 u_1 & \rho u_1 u_2 & \rho u_1 u_3 \\ \rho u_2 u_0 & \rho u_2 u_1 & \rho u_2 u_2 & \rho u_2 u_3 \\ \rho u_3 u_0 & \rho u_3 u_1 & \rho u_3 u_2 & \rho u_3 u_3 \end{bmatrix} \tag{201}$$

...with u_μ and u_ν as defined in equations (25) to (28).

This tensor may be written in a more explicit form, using the "traditional" velocity v_x, v_y and v_z instead of the relativistic velocities, as shown in equations (25) to (28):

$$T_{\mu\nu} = \begin{bmatrix} \rho\gamma^2 c^2 & \rho\gamma^2 c v_x & \rho\gamma^2 c v_y & \rho\gamma^2 c v_z \\ \rho\gamma^2 c v_x & \rho\gamma^2 v_x v_x & \rho\gamma^2 v_x v_y & \rho\gamma^2 v_x v_z \\ \rho\gamma^2 c v_y & \rho\gamma^2 v_y v_x & \rho\gamma^2 v_y v_y & \rho\gamma^2 v_y v_z \\ \rho\gamma^2 c v_z & \rho\gamma^2 v_z v_x & \rho\gamma^2 v_z v_y & \rho\gamma^2 v_z v_z \end{bmatrix} \tag{202}$$

For low velocities, we have $\gamma = 1$:

$$T_{\mu\nu} = \begin{bmatrix} \rho c^2 & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \rho v_x v_x & \rho v_x v_y & \rho v_x v_z \\ \rho c v_y & \rho v_y v_x & \rho v_y v_y & \rho v_y v_z \\ \rho c v_z & \rho v_z v_x & \rho v_z v_y & \rho v_z v_z \end{bmatrix} \tag{203}$$

Replacing the spatial part of this tensor by the old definitions (equation 189) gives:

$$T_{\mu\nu} = \begin{bmatrix} \rho c^2 & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \sigma_x & \tau_{xy} & \tau_{xz} \\ \rho c v_y & \tau_{yx} & \sigma_y & \tau_{yz} \\ \rho c v_z & \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{204}$$

Replacing ρ by m/V (equation 193) and mc^2 by E leads to one of the most commons form of the Energy-Momentum Tensor, V being the volume:

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} E/V & \rho c v_x & \rho c v_y & \rho c v_z \\ \rho c v_x & \sigma_x & \tau_{xy} & \tau_{xz} \\ \rho c v_y & \tau_{yx} & \sigma_y & \tau_{yz} \\ \rho c v_z & \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{205}$$

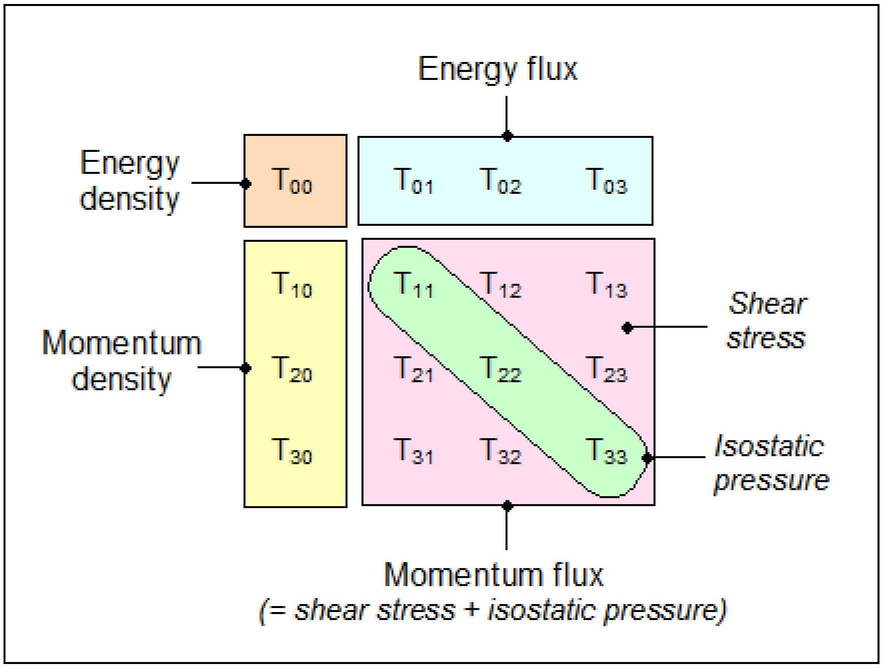

Figure 6: Meaning of the Energy-Momentum Tensor.

9 - Einstein Field Equations

Finally, the association of the Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R$ (equation 134), the Einstein Constant $8\pi G_0/c^4$ (equation 161), and the Energy-Momentum Tensor $T_{\mu\nu}$ (equations 204/205), gives the full Einstein Field Equations, excluding the cosmological constant Λ which is not proven:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_0}{c^4} T_{\mu\nu}} \tag{206}$$

To summarize, the Einstein field equations are 16 nonlinear partial differential equations that describe the curvature of spacetime, i.e. the gravitational field, produced by a given mass. As a result of the symmetry of $G_{\mu\nu}$ and $T_{\mu\nu}$, the actual number of equations are reduced to 10, although there are an additional four differential identities (the Bianchi identities) satisfied by $G_{\mu\nu}$, one for each coordinate.

10 - Dimensional Analysis

The dimensional analysis verifies the homogeneity of equations. The most common dimensional quantities used in this chapter are:

Speed V $\Rightarrow [L/T]$
Energy E $\Rightarrow [ML^2/T^2]$
Force F $\Rightarrow [ML/T^2]$
Pressure P $\Rightarrow [M/LT^2]$
Momentum M $\Rightarrow [ML/T]$
Gravitation constant G $\Rightarrow [L^3/MT^2]$

Here are the dimensional analysis of the Energy-Momentum Tensor:

$\mathbf{T_{00}}$ is the energy density, i.e. the amount of energy stored in a given region of space per unit volume. The dimensional quantity of E is $[ML^2/T^2]$ and V is $[L^3]$. So, the dimensional quantity of E/V is $[M/LT^2]$. Energy density has the same physical units as pressure which is $[M/LT^2]$.

$\mathbf{T_{01}}, \mathbf{T_{02}}, \mathbf{T_{03}}$ are the energy flux, i.e. the rate of transfer of energy through a surface. The quantity is defined in different ways, depending on the context. Here, ρ is the density $[M/V]$, and c and v_i (i = 1 to 3) are velocities $[L/T]$. So, the dimensional quantity of T_{0i} is $[M/L^3][L/T][L/T]$. It is that of a pressure $[M/LT^2]$.

$\mathbf{T_{10}}, \mathbf{T_{20}}, \mathbf{T_{30}}$ are the momentum density, which is the momentum per unit volume. The dimensional quantity of T_{i0} (i = 1 to 3) is identical to T_{0i}, i.e. that of a pressure $[M/LT^2]$.

$\mathbf{T_{12}}, \mathbf{T_{13}}, \mathbf{T_{23}}, \mathbf{T_{21}}, \mathbf{T_{31}}, \mathbf{T_{32}}$ are the shear stress, or a pressure $[M/LT^2]$.

$\mathbf{T_{11}}, \mathbf{T_{22}}, \mathbf{T_{33}}$ are the normal stress or isostatic pressure $[M/LT^2]$.

Note: The Momentum flux is the sum of the shear stresses and the normal stresses.

As we see, all the components of the Energy-Momentum Tensor have a pressure-like dimensional quantity:

$$\boxed{T_{\mu\nu} \Rightarrow \text{Pressure } [M/LT^2]} \tag{207}$$